Localization of the Eigenvalues of Linear Integral Equations with Applications to Linear Ordinary Differential Equations

by

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Introduction: In this paper we consider integral equations of the form:

$$\varphi(P) = \lambda \int_D K(P,Q)\varphi(Q)dQ + f(Q) \tag{1}$$

where D is a bounded measurable set of R^d, Euclidean space of dimension d, $K(P,Q)$ is defined and square integrable over $D \times D$ (not necessarily symmetric), f is defined and square integrable over D and φ is the unknown function.

It is our purpose (i) to show that the eigenvalues of (1) can be approximated by the reciprocal eigenvalues of a finite matrix $K^m = (K_{ij})$ of order m that is easily determined from K, (ii) to derive an explicit error estimate for the approximation that depends on K and m and (iii) to apply the results of (i) and (ii) to the problem of effectively approximating the eigenvalues of a self adjoint ordinary differential equation.

Let us form, for some complete orthonormal set in $L_2(D)$, the matrix (K_{ij}) of Fourier coefficients of K. We shall show (Theorem 1) that the problem of finding the eigenvalues and eigenfunctions of (1) is equivalent to the problem of finding the eigenvalues and eigenvectors of the infinite matrix (K_{ij}). Having done this, we shall prove (Theorem 2) a generalization of the Gerschgorin circle theorem that applies to the infinite matrix (K_{ij}).

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In theorem 3, we shall consider the case when an upper left hand square matrix K^m of (K_{ij}) is similar to a diagonal matrix and show that the eigenvalues of (K_{ij}) lie in the union of circles centered at zero and the eigenvalues of A, with radii depending only on m, those K_{ij} for $1 \le i \le m$, $j \ge m + 1$, and the norm of the matrix used to diagonalize K^m. Moreover, exactly as many eigenvalues, counting multiplicities, lie in each connected component as circles that make up the component. When the matrix K^m is symmetric, the radii depend only on m and K_{ij}, $1 \le i \le m$, $j \ge m + 1$.

In the event the radii diminish with m, the explicit bounds on the radii can be used to give error bounds on how closely the eigenvalues of the upper left matrix approximate the reciprocal eigenvalues of (1). Thus the problem of finding eigenvalues of (1) to a prescribed degree of accuracy, reduces to choosing m judiciously and then computing the eigenvalues of the $m \times m$ matrix.

We apply the results to self adjoint ordinary differential equations. In this case K becomes the Green's function (or a minor variant of it) and the eigenvalues of the differential equation are those of the integral equation. Using the complex exponentials as the orthonormal set, the matrix (K_{ij}), whose reciprocal eigenvalues are the eigenvalues of the differential equation, is made up of the Fourier coefficients of the Green's function, moreover (K_{ij}) is symmetric. The error estimate can be written as $Ar_m(n)$, where n is the order of the differential equation, A is a constant depending on the Green's function and the differential equation and

$$r_m(n) = O\left(\frac{1}{m^{n-3/2}}\right) . \tag{2}$$

The error estimate for second order equations, though of theoretical interest, is of little practical value. For fourth and higher order differential equations the estimate, because of (2), proves to be of great practical value. By consulting a table for $r_m(n)$ and computing A, the size of the matrix (K_{ij}), that will give desired accuracy, can be determined by inspection. A table for $r_m(2)$ and $r_m(4)$ is given.

As specific illustrations of the method, we choose the fourth order equation governing the transverse displacement of a whirling shaft fixed to rotate between ball bearings at each end, see [7, p.442]. The eigenvalues for this problem are known. In order to illustrate how the method can be adapted to second order equations, we chose the equation governing the motion of an inhomogeneous string, fixed at one end and restricted to move transversally by an elastic force at the other end. We convert this problem to an equivalent fourth order differential system, and apply our method to approximate the eigenvalues. The results are compared with those of Collatz [3, p.257] who has bracketed the first few eigenvalues.

For references to the literature, see e.g., [1], [2], [3], [5], [6], [8], [9], [10]. The method developed in this paper, as applied to differential equations, is dependent on finding the Green's function and thus its use is more restrictive than the variational methods. However, because of the error estimate, the problem of finding an upper bound for an eigenvalue and the problem of finding a lower bound for an eigenvalue (in general a much more difficult problem) are solved simultaneously, and to any preassigned accuracy.

Since working out the results of this paper, we have come across the results of Lösch [6], who proved that the eigenvalues of (K_{ij}) converge to the reciprocal eigenvalues of the integral equation with kernel K. However in his paper he did not derive an error estimate.

We would like to express our deep gratitude to Charles Akemann for carrying out the numerical computations involved in finding the matrices

$(K_{ij})^m$ and the corresponding eigenvalues used in the two illustrations.

§1. Equivalence. In this paragraph we prove the equivalence of the given integral equation with an infinite system of linear equations. Before proving the theorem we prove two lemmas.

Lemma 1. *If* $K(P,Q)$ *is a complex valued function in* $L_2(G \times G)$, $\{\alpha_i(P): i=1,2,\ldots\}$, *is a complete complex-valued orthonormal set in* $L_2(G)$, *and* $\varphi(P) \in L_2(G)$ *then for*

$$\beta_i(Q) = \int_G K(P,Q)\alpha_i(P)dP$$

$$(1.1) \qquad K(P,Q)\varphi(Q), \quad \varphi(Q)\sum_{j=1}^{\infty} \overline{\alpha_j(P)}\,\beta_j(Q) \in L_1(G) \quad \textit{for almost all } P,$$

$$(1.1.1) \qquad K(P,Q)\varphi(Q), \quad \varphi(Q)\sum_{j=1}^{\infty} \overline{\alpha_j(P)}\,\beta_j(Q) \in L_1(G) \quad \textit{for almost all } Q,$$

and

$$(1.2) \qquad \int_G K(P,Q)\varphi(Q)dQ = \int_G \varphi(Q)\sum_{j=1}^{\infty} \overline{\alpha_j(P)}\,\beta_j(Q)dQ \quad \textit{for almost all } P,$$

$$(1.2.1) \qquad \int_G K(P,Q)dP = \int_G \sum_{j=1}^{\infty} \overline{\alpha_j(P)}\,\beta_j(Q)dP \quad \textit{for almost all } Q.$$

Proof. To show (1.1) and (1.1.1) it suffices to show that $K(P,Q)$ and

$$(1.3) \qquad \sum_{j=1}^{\infty} \overline{\alpha_j(P)}\,\beta_j(Q)$$

are equivalent $L_2(G \times G)$ functions since then by Schwarz's inequality:

$$\int_G \int_G |K(P,Q)\varphi(Q)|\,dQ\,dP \le \int_G \int_G |K(P,Q)|^2\,dP\,dQ \int_G |\varphi(P)|^2 dP \text{ measure } G$$

and thus by Fubini's theorem $K(P,Q)\varphi(Q) \in L_1(G)$ for almost all P and in $L_1(G)$ for almost all Q. Similarly (1.2) will follow upon showing $K(P,Q)$ and (1.3) are equivalent $L_2(G \times G)$ functions.

Note

$$h_n(Q) \equiv \int_G |K(P,Q) - \sum_{i=1}^{n} \overline{\alpha_i(P)}\,\beta_i(Q)|^2\,dP \to 0 \quad \text{as} \quad n \to \infty, \quad \text{for almost all } Q \text{ in } G,$$

since $\{\alpha_i(P)\}$ is a complete set in $L_2(G)$. Thus given any $\epsilon > 0$, by Egorov's theorem, there exists a measurable subset $\tilde{G}$ of G such that

$$\int_{\tilde{G}} h_n(Q)\,dQ < \frac{\epsilon}{2}$$

and the measure of $G/\tilde{G} < \dfrac{\epsilon}{8\|K\|_2^2}$. Also

$$\int_G |K(P,Q) - \sum_{i=1}^{n} \overline{\alpha_i(P)}\,\beta_i(Q)|^2 dP \le 2\Big[\int_G |K(P,Q)|^2 dP + \int_G |\sum_{i=1}^{n} \overline{\alpha_i(P)}\,\beta_i(Q)|^2 dP\Big]$$

$$\le 2\Big[\int_G |K(P,Q)|^2 dP + \sum_{i=1}^{n} |\beta_i(Q)^2\Big]$$

$$\le 4\int_G |K(P,Q)|^2\,dP$$

where use has been made of the orthonormality of α_i and Bessel's inequality. Thus

$$\|K(P,Q) - \sum_{i=1}^{n} \overline{\alpha_i(P)}\beta_i(Q)\|_2^2 = \int h_n(Q)dQ = \Big(\int_{G/\tilde{G}} + \int_{\tilde{G}}\Big) h_n(Q)dQ$$

$$\le \frac{\epsilon}{8\|K\|_2^2} \cdot 4\|K\|_2^2 + \frac{\epsilon}{2} = \epsilon$$

which proves the assertion.

Next we prove

<u>Lemma</u> 2. <u>Let</u> $K(P,Q) \in L_2(G \times G)$, $\alpha_i(P)$ <u>and</u> $\beta_i(P)$ <u>be as in Lemma</u> 1. <u>Then</u>

$$\sum_{i=1}^{\infty} \overline{\alpha_i(P)}\,\beta_i(Q) \in L_1(G)$$ <u>for almost all</u> P <u>in</u> G <u>and</u>

(i) <u>if</u> $\varphi(P) \in L_2(G)$ <u>then for almost all</u> P <u>in</u> G

$$\int_G \sum_{i=1}^{\infty} \overline{\alpha_i(P)}\,\beta_i(Q)\,\varphi(Q)dQ = \sum_{i=1}^{\infty} \overline{\alpha_i(P)} \int_G \beta_i(Q)\,\varphi(Q)dQ.$$

<u>Moreover, for</u> $C_i = \int_G \varphi(P)\beta_i(P)dP$ <u>we have</u>

$$\sum_{k=1}^{\infty} C_k\,\overline{\alpha_k(Q)} \in L_1(G).$$ <u>Similarly</u>

(ii) <u>if</u> $C = (C_1, C_2, \ldots) \in \ell_2$ <u>then</u>

$$\int_G \sum_{k=1}^{\infty} C_k\,\overline{\alpha_k(Q)}\,\beta_j(Q)dQ = \sum_{k=1}^{\infty} C_k \int_G \beta_j(Q)\,\overline{\alpha_k(Q)}\,dQ.$$

<u>Proof</u>: Note that by lemma 1, Schwarz's inequality and compactness

$$\sum_{i=1}^{\infty} \overline{\alpha_i(P)}\,\beta_i(Q) \in L_1(G \times G).$$

Applying Fubini's theorem we obtain

$$\sum_{i=1}^{\infty} \overline{\alpha_i(P)}\,\beta_i(Q) \in L_1(G) \quad \text{for almost all } P.$$

By the Riesz-Fischer theorem we obtain immediately that

$$\sum_{k=1}^{\infty} C_k \overline{\alpha_k(Q)} \in L_2(G).$$

and by Schwarz's lemma it is also in $L_1(G)$.

(i) Let

$$S(P,Q) = \sum_{i=1}^{\infty} \overline{\alpha_i(P)}\ \beta_i(Q)\ \varphi(Q)$$

and

$$S_n(P,Q) = \sum_{i=1}^{n} \overline{\alpha_i(P)}\ \beta_i(Q)\ \varphi(Q) \qquad (n = 1,2,\ldots).$$

First note that

$$\int_G \int_G |S(P,Q) - S_n(P,Q)|\,dQ\ dP \le |G|^2 \int_G \int_G |S(P,Q) - S_n(P,Q)|^2\,dQ\ dP,$$

where $|G|$ = measure of G. Now

$$\int_G \int_G |S(P,Q) - S_n(P,Q)|^2 dQ\ dP \le \sum_{k=n+1}^{\infty} \|\beta_k\|_2^2\ \|\varphi\|_2^2 .$$

Here use has been made of Schwarz's lemma, and Parseval's equality. Since $\sum\limits_{k=1}^{\infty} \|\beta_k\|_2^2 = \|K\|_2^2$ we see that $\lim \sum\limits_{k=n}^{\infty} \|\beta_k\|_2^2 = 0$ and thus

$$\lim_n \int_G \int_G |S(P,Q) - S_n(P,Q)|^2 dQ\ dP = 0.$$

Applying Fatou's lemma we obtain

$$\int_G [\lim_n \int_G |S(P,Q) - S_n(P,Q)|\,dQ]\,dP = 0$$

and hence

$$\lim_n \int_G |S(P,Q) - S_n(P,Q)|\,dQ = 0 \quad \text{for almost all } P \text{ in } G.$$

But this clearly implies the conclusion.

(ii) Similarly, since

$$\left|\int_G \sum_{k=n}^{\infty} c_k \overline{\alpha_k(Q)}\ \beta_j(Q)\, dQ\right|^2 \le \int_G |\beta_j(Q)|^2\, dQ \int_G \left|\sum_{k=n}^{\infty} c_k \overline{\alpha_k(Q)}\right|^2 dQ$$

$$\le \|K\|_2^2 \sum_{k=n}^{\infty} |c_k|^2$$

where Schwarz's inequality, Bessel's inequality and Parseval's equality have been used. The result follows.

Now we are in a position to prove the equivalence theorem viz:

<u>Theorem</u> 1. (<u>Equivalence of integral equation with infinite system of linear equations</u>) <u>Let</u> $\{\alpha_j(P)\colon j = 1,2,\ldots\}$ <u>be a complete orthonormal set in</u> $L_2(G)$ <u>and let</u> $K(P,Q) \in L_2[G \times G]$. <u>Consider the integral equation</u>

(1.4) $$\psi(P) = \mu \int_G K(P,Q)\psi(Q)dQ.$$

<u>Let</u> λ <u>be an eigenvalue and</u> $\varphi(P)$ <u>be a corresponding function in</u> $L_2(G)$ <u>of</u> (1.4). <u>Let</u>

(1.5) $$\beta_i(Q) = \int_G K(P,Q)\ \alpha_i(P)\, dP,$$

(1.6) $$K_{ij} = \int_G \int_G K(P,Q)\ \alpha_i(P)\ \overline{\alpha_j(Q)}\ dP\, dQ$$

(1.7) $$c_i = \int \varphi(P)\beta_i(P)\, dP .$$

Then λ is an eigenvalue for the infinite system

$$(1.8) \qquad x_i = \mu \sum_{j=1}^{\infty} K_{ij} x_j$$

and $C = (C_1, C_2, \ldots)$ is a corresponding eigenvector in ℓ_2.

Conversely, if λ is an eigenvalue of (1.8) and $C = (C_1, C_2, C_3, \ldots)$ in ℓ_2, is a corresponding eigenvector and

$$(1.9) \qquad \varphi(P) = \lambda \sum_{i=1}^{\infty} C_i \overline{\alpha_i(P)}$$

then λ is an eigenvalue for (1.4) and $\varphi(P)$ is a corresponding eigenfunction in $L_2(G)$. Moreover

$$(1.10) \qquad \|\varphi\| = |\lambda| \, \|C\| .$$

Proof: Assume λ is an eigenvalue and $\varphi(P)$ is a corresponding eigenfunction of (1.4) which is in $L_2(G)$. By lemma 1, we obtain for almost all P in G

$$\varphi(P) = \lambda \int_G \varphi(Q) \sum_{j=1}^{\infty} \overline{\alpha_j(P)} \, \beta_j(Q) \, dQ$$

$$= \lambda \sum_{j=1}^{\infty} \overline{\alpha_j(P)} \int_G \varphi(Q) \beta_j(Q) dQ \qquad \text{by lemma 2}$$

$$= \lambda \sum_{j=1}^{\infty} \overline{\alpha_j(P)} \, C_j \qquad \text{by (1.7).}$$

Substituting back into (1.4) for φ we obtain

$$(1.11)\qquad \lambda \sum_{j=1}^{\infty} \overline{\alpha_j(P)}\, C_j = \lambda \sum_{j=1}^{\infty} \overline{\alpha_j(P)} \int_G \beta_j(Q) \Big[\lambda \sum_{k=1}^{\infty} \overline{\alpha_k(Q)}\, C_k \Big] dQ$$

$$= \lambda \sum_{j=1}^{\infty} \overline{\alpha_j(P)} \Big[\lambda \sum_{k=1}^{\infty} C_k \int_G \beta_j(Q)\, \overline{\alpha_k(Q)}\, dQ \Big] \quad \text{by lemma 2.}$$

But $\int_G \beta_j(Q)\, \overline{\alpha_k(Q)}\, dQ = \int_G \int_G K(R,Q)\, \alpha_j(R)\, \overline{\alpha_k(Q)}\, dR\, dQ = K_{jk}$. Thus since the $\alpha_i(P)$'s are linearly independent the λ and C_i satisfy (1.8).

Next note that

$$\sum_{j=1}^{\infty} |C_j|^2 = \sum_{j=1}^{\infty} \Big| \int \varphi(P) \beta_j(P)\, dP \Big|^2 \le \|\varphi\|^2 \sum_{j=1}^{\infty} \|\beta_j\|^2 \le \|\varphi\|^2\, \|K\|^2$$

where use has been made of Bessel's inequality. Thus C is in ℓ_2 and the first part of the theorem is established.

The argument is reversible and thus if λ is an eigenvalue and C a corresponding eigenvector in ℓ_2, of (1.8), then φ defined by (1.9) is an eigenfunction corresponding to the eigenvalue λ of (1.4). We showed in lemma 2, that φ, as defined by (1.9), is in $L_2(G)$. But since the α_i are orthonormal

$$\|\varphi\|^2 = |\lambda|^2 \sum_{j=1}^{\infty} |C_j|^2 .$$

§2. Localization of eigenvalues. Because of the equivalence of the integral equation with the infinite linear system, expressed in theorem 1, it suffices to study the location of the eigenvalues of the infinite system. The localization of these eigenvalues is expressed in theorems 2 and 3.

Theorem 2. Consider the infinite system

$$(1.8)\qquad \lambda x_i = \sum_{j=1}^{\infty} K_{ij}x_j, \quad \text{where} \quad \sum_{i,j=1}^{\infty} |K_{ij}|^2 < \infty .$$

Given any $\epsilon_1, \epsilon_2 > 0$ choose m and q such that

$$(2.1)\qquad \sum_{i=m+1}^{\infty} \sum_{j=1}^{\infty} |K_{ij}|^2 < \frac{\epsilon_1^2}{2} \quad \text{and} \quad \sum_{j=q+1}^{\infty} \sum_{i=1}^{\infty} |K_{ij}|^2 < \frac{\epsilon_2^2}{2},$$

and let

$$\rho_k^2 = 2m \sum_{j \neq k} |K_{kj}|^2, \qquad 1 \leq k \leq m$$

and

$$\gamma_j^2 = 2q \sum_{k \neq j} |K_{kj}|^2, \qquad 1 \leq j \leq q.$$

Let λ be an eigenvalue of (1.8) and $C = (C_1, C_2, \ldots)$ be a corresponding eigenvector in ℓ_2. Then λ lies in

$$(2.2)\qquad \bigcup_{k=1}^{m} \{z\colon |z - K_{kk}| < \rho_k\} \cup \{z\colon |z| < \epsilon_1\}$$

and in

$$(2.3)\qquad \bigcup_{j=1}^{q} \{z\colon |z - K_{jj}| < \gamma_j\} \cup \{z\colon |z| < \epsilon_2\}$$

Moreover, each component of (2.2) (resp (2.3)) contains exactly as many eigenvalues as circles, where the eigenvalues and circles are counted with their multiplicities.

In the case (1.8) corresponds to the integral equation (1.4), then (2.1) becomes

$$(2.1.1)\qquad \sum_{i=m+1}^{\infty} \|\beta_i\|_2^2 < \frac{\epsilon_1^2}{2}, \quad \sum_{j=q+1}^{\infty} \|\beta_j\|_2^2 < \frac{\epsilon_2^2}{2}.$$

<u>Proof</u>: Let λ be an eigenvalue of (1.5) (without loss of generality assume $\lambda \neq 0$) and $C = (C_1, C_2, \ldots)$ be a corresponding eigenvector of unit length. Then

$$(\lambda - K_{kk})C_k = \sum_{j \neq k} K_{kj} C_j. \qquad (k = 1,2,3,\ldots) \tag{2.4}$$

Note that

$$\begin{aligned}
\sum_{j=m+1}^{\infty} |C_j|^2 &\le \Big[\sum_{j=m+1}^{\infty} |C_j| \Big]^2 \\
&\le \Big[\sum_{j=m+1}^{\infty} |\lambda^{-1}|^2 \, \Big| \sum_{k=1}^{\infty} K_{jk} C_k \Big|^2 \Big] \\
&\le |\lambda^{-1}|^2 \sum_{j=m+1}^{\infty} \Big[\sum_{k=1}^{\infty} |K_{jk}|^2 \, \|C\|_2^2 \Big] \\
&\le |\lambda^{-1}|^2 \sum_{j=m+1}^{\infty} \sum_{k=1}^{\infty} |K_{jk}|^2.
\end{aligned} \tag{2.5}$$

If $|\lambda| < \epsilon$ the result is clear. Assume that $|\lambda| \ge \epsilon$. Then by (2.5) and (2.1)

$$\sum_{j=m+1}^{\infty} |C_j|^2 \le \epsilon^{-2} \sum_{j=m+1}^{\infty} \sum_{k=1}^{\infty} |K_{jk}|^2 < \frac{1}{2} \tag{2.6}$$

and there exists $1 \le \ell \le m$ such that $|C_\ell|^2 > (2m)^{-1}$. Hence (2.4) becomes

$$\begin{aligned}
|\lambda - K_{\ell\ell}|^2 &\le |C_\ell|^{-2} \sum_{j \neq \ell} |K_{\ell j}|^2 \sum_{j \neq \ell} |C_j|^2 \\
&\le |C_\ell|^{-2} [1 - |C_\ell|^2] \sum_{j \neq \ell} |K_{\ell j}|^2 \\
&\le 2m \sum_{j \neq \ell} |K_{\ell j}|^2.
\end{aligned}$$

The conclusion (2.2) follows immediately. (2.3) follows analogously and the argument involving components follows exactly as in the case of Gerschgorin circles.

Consider now the integral equation (1.4) and the corresponding system (1.8). By the definition (1.5) of $\beta_i(Q)$, it is clear from Schwarz's lemma that $\beta_i(Q) \in L_2(G)$ and thus it has an expansion

$$\beta_i(Q) = \sum_{k=1}^{\infty} K_{ik}\, \alpha_k(Q)$$

where the K_{ik}'s are given in (1.6). Thus

$$\sum_{i=m+1}^{\infty} \|\beta_i\|_2^2 = \sum_{i=m+1}^{\infty} \int_G |\beta_i(Q)|^2\, dQ$$

$$= \sum_{i=m+1}^{\infty} \sum_{k=1}^{\infty} |K_{ik}|^2$$

by Parseval's equality, which concludes the proof. (2.3) follows similarly.

Next we shall apply the results of theorem 2 to a particularly important special case, i.e., when the upper left matrix of the infinite system is similar to a diagonal matrix. These results are given in:

Theorem 3. *Consider the infinite system*

(2.7) $$\lambda x_i = \sum_{j=1}^{\infty} K_{ij} x_j, \qquad K_{ij} \text{ is in general complex.}$$

Let T *be an* $m \times m$ *matrix for which*

$$T^{-1} K^m T = F = \begin{bmatrix} f_1 & & & 0 \\ & f_2 & & \\ & & \ddots & \\ 0 & & & f_m \end{bmatrix} \qquad (T \text{ is in general complex})$$

where

$$K^m = (K_{ij})^{m\times m}\,, \qquad\qquad 1 \le i,j \le m$$

and

$$(2.8) \qquad \sum_{j=1}^{m} |t_{jk}|^2 \le t_0^2\,, \qquad \sum_{k=1}^{m} |\tau_{jk}|^2 \le t_0^2, \qquad t_0^2 m > 1,$$

where $T = (t_{jk})$, $T^{-1} = (\tau_{jk})$. Let

$$p_m^2 = t_0^2 \sum_{j=m+1}^{\infty} \sum_{\ell=1}^{m} |K_{\ell j}|^2, \qquad r_m^2 = 2m\, p_m^2, \qquad r_0^2 = 2m\, p_m^2$$

$$q_m^2 = t_0^2 \sum_{k=m+1}^{\infty} \sum_{\ell=1}^{m} |K_{k\ell}|^2, \qquad s_m^2 = 2m\, q_m^2, \qquad s_0^2 = 2m\, q_m^2\,.$$

If λ is an eigenvalue of (2.7), and $(c_1, c_2, \ldots)$ a corresponding eigenvector in ℓ_2, then λ lies in

$$(2.9) \qquad \bigcup_{k=1}^{m} \{z\colon |z - f_k| < r_m\} \cup \{z\colon |z| < r_0\}$$

and in

$$(2.10) \qquad \bigcup_{j=1}^{m} \{z\colon |z - f_k| < s_m\} \cup \{z\colon |z| < s_0\}.$$

Moreover, each component of (2.9) (respectively (2.10)) contains exactly as many eigenvalues as circles where the eigenvalues and circles are counted with their multiplicities.

Proof: Let

$$B = (K_{ij}) \qquad 1 \le i \le m, \qquad m + 1 \le j$$

$$C = (K_{ij}) \qquad m + 1 \le i, \qquad 1 \le j \le m$$

$$D = (K_{ij}) \qquad m + 1 \le i, \qquad m + 1 \le j$$

$$\tilde{K} \equiv (\tilde{K}_{ij}) \equiv \begin{bmatrix} T^{-1} & O \\ O & I \end{bmatrix} \left(K_{ij} \right) \begin{bmatrix} T & O \\ O & I \end{bmatrix} = \begin{bmatrix} F & T^{-1}B \\ CT & D \end{bmatrix} .$$

Note that $\tilde{K}$ has the same eigenvalues as K. If we let

$$(2.11) \qquad \tilde{\rho}_k^2 = 2m \sum_{j \neq k} |\tilde{K}_{kj}|^2 = 2m \sum_{j=m+1}^{\infty} |\tilde{K}_{kj}|^2, \qquad 1 \le k \le m,$$

and

$$(2.12) \qquad \tilde{\gamma}_j^2 = 2m \sum_{k \neq j} |\tilde{K}_{kj}|^2 = 2m \sum_{k=m+1}^{\infty} |\tilde{K}_{kj}|^2, \qquad 1 \le j \le m,$$

Then we can conclude from theorem 2 that the eigenvalues of K lie in

$$(2.13) \qquad \bigcup_{k=1}^{m} \{z: |z - f_k| < \tilde{\rho}_k\} \cup \{z: |z| < \epsilon_1\}$$

and in

$$(2.14) \qquad \bigcup_{j=1}^{m} \{z: |z - f_j| < \tilde{\gamma}_j\} \cup \{z: |z| < \epsilon_2\}$$

where

$$(2.15) \qquad \sum_{i=m+1}^{\infty} \sum_{j=1}^{\infty} |\tilde{K}_{ij}|^2 < \frac{\epsilon_1^2}{2}, \qquad \sum_{j=q+1}^{\infty} \sum_{i=1}^{\infty} |\tilde{K}_{ij}|^2 < \frac{\epsilon_2^2}{2} .$$

But for $1 \le k \le m$, $m+1 \le j$

$$(2.16) \qquad |\tilde{K}_{kj}|^2 = |(T^{-1}B)_{kj}|^2 = \Big| \sum_{\ell=1}^{m} \tau_{k\ell} K_{\ell j} \Big|^2,$$

$$\le \sum_{\ell=1}^{m} |\tau_{k\ell}|^2 \sum_{\ell=1}^{m} |K_{\ell j}|^2$$

$$\le t_0^2 \sum_{\ell=1}^{m} |K_{\ell j}|^2 .$$

Thus combining (2.11) and (2.16) gives

$$(2.17) \qquad \tilde{\rho}_k^2 \le 2m \quad t_0^2 \sum_{j=m+1}^{\infty} \sum_{\ell=1}^{m} |K_{\ell j}|^2 = r_m^2, \qquad 1 \le k \le m,$$

and similarly

$$(2.18) \qquad \tilde{\gamma}_j^2 \le 2m \quad t_0^2 \sum_{k=m+1}^{\infty} \sum_{\ell=1}^{m} |K_{k\ell}|^2 = s_m^2, \qquad 1 \le j \le m.$$

Finally we note that

$$
\begin{aligned}
(2.19) \quad \sum_{i=m+1}^{\infty} \sum_{j=1}^{\infty} |\tilde{K}_{ij}|^2 &= \sum_{i=m+1}^{\infty} \sum_{j=m+1}^{\infty} |K_{ij}|^2 + \sum_{i=m+1}^{\infty} \sum_{j=1}^{m} |\tilde{K}_{ij}|^2 \\
&= \sum_{i=m+1}^{\infty} \sum_{j=m+1}^{\infty} |K_{ij}|^2 + \sum_{i=m+1}^{\infty} \sum_{j=1}^{m} |(CT)_{ij}|^2 \\
&\le \sum_{i=m+1}^{\infty} \sum_{j=m+1}^{\infty} |K_{ij}|^2 + \sum_{i=m+1}^{\infty} \Big[\sum_{j=1}^{m} | \sum_{\ell=1}^{m} K_{i\ell} t_{\ell j}|^2 \Big] \\
&\le \sum_{i=m+1}^{\infty} \sum_{j=m+1}^{\infty} |K_{ij}|^2 + \sum_{i=m+1}^{\infty} \Big[\sum_{j=1}^{m} \Big\{ \sum_{\ell=1}^{m} |K_{i\ell}|^2 \Big\} \Big\{ \sum_{\ell=1}^{m} |t_{\ell j}|^2 \Big] \\
&\le \sum_{i=m+1}^{\infty} \Big[\sum_{j=m+1}^{\infty} |K_{ij}|^2 + m t_0^2 \sum_{\ell=1}^{m} |K_{i\ell}|^2 \Big] \\
&< m t_0^2 \sum_{i=m+1}^{\infty} \sum_{j=1}^{\infty} |K_{ij}|^2 \quad \text{since} \quad m t_0^2 > 1 \\
&= s_0^2 \big/ 2 .
\end{aligned}
$$

Thus for the ϵ_1 of (2.15) we can take s_0. Combining (2.13), (2.14), (2.17), (2.18) and (2.19) we obtain the conclusion (2.9) and similarly (2.10).

Corollary. In the event that K^m is Hermitian ~~symmetric~~ we can take $t_0 = 1$.

Proof. This follows, since for T we can take an orthogonal matrix, thus each row and column has Euclidean length one and $T' = T^{-1}$.

§3. Application to ordinary differential equations. Consider the ordinary differential equation of degree $n \geq 2$

$$L[u] + \lambda u = r(x) \tag{3.1}$$

on the interval $[0, \ell]$ with the system of boundary conditions

$$M_i[u] = A_i[u] + B_i[u], \qquad i = 1,2,\dots,n, \tag{3.2}$$

where

$$L[u] = p_n(x)u^{(n)}(x) + p_{n-1}(x)u^{(n-1)}(x) + \cdots + p_0(x)u(x) = 0, \quad p_n \neq 0,$$

and $A_i[u]$ are boundary conditions relative to the end point 0 and $B_i[u]$ are relative to the end point ℓ. Let $G(x,\xi)$ be the Green's function for this system. Then, as is well known, see e.g. [4], if the system is self adjoint then G is symmetric, $G(x,\xi) \in C^{n-2}[0,\ell]$,

$$D_x^{n-1} G(\xi+0, \xi) - D_x^{n-1} G(\xi-0, \xi) = 1/p_n(\xi)$$

and (3.1) and (3.2) are equivalent to the integral equation

$$u(x) + \lambda \int_0^\ell G(x,\xi)u(\xi)d\xi = f(x)$$

where

$$f(x) = \int_0^\ell G(x,\xi)r(\xi)d\xi.$$

In the event we take the orthonormal set to be

$$\ell^{-\frac{1}{2}} e^{i \frac{2\pi}{\ell} \mu s}, \quad \mu = 0,1,2,\dots,.$$

we can state the following

<u>Lemma</u> 3.1. <u>If</u> $G(x,\xi)$ <u>is the Green's function of</u> $p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_0(x)y(x) = 0$ <u>on</u> $[0,\ell]$ <u>with boundary conditions of the form</u> (3.2), <u>and</u> $K(x,\xi) = \rho(x)\rho(\xi)G(x,\xi)$ <u>is the kernel of the equivalent integral equation</u> $\rho \in C^n[0,\ell]$ $\rho > 0$ on $[0,\ell]$ <u>and</u>

$$g_{\mu\nu}(K) = \frac{1}{\ell}\int_0^\ell ds \int_0^\ell dt\ K(s,t)e^{+\frac{2\pi i}{\ell}(\mu s - \nu t)} ;$$

<u>then</u>

$$|g_{\mu\nu}| \le A\,\nu^{-n}, \qquad \mu \ge 0, \quad \nu > 0,$$

<u>where</u>

$$A = \ell^{-1}\left(\frac{\ell}{2\pi}\right)^n C \tag{3.3}$$

$$C = \int_0^\ell [\,|D_t^{n-1}K(s,0)| + |D_t^{n-1}K(s,\ell)| + 2\ell \sup_t B(s,t)]ds \tag{3.4}$$

$$B(s,t) = |\rho(s)| \sum_{k=1}^{n}\left[\left|\binom{n}{k}D_t^k\rho(t) - \rho(t)p_n^{-1}(t)p_{n-k}(t)\right| + \left|\binom{n-1}{k-1}D_t^{k-1}\rho(t)\right|\right]|D_t^{n-k}G(s,t)|. \tag{3.5}$$

<u>Proof</u>. Let $\rho_\varepsilon(x,\xi)$ be the $C^\infty(R_2)$ - function that is 1 for $\varepsilon < x < \ell-\varepsilon$, $\varepsilon < \xi < \ell-\varepsilon$ and has support in the square $[0,\ell] \times [0,\ell]$. If we let

$$K^\varepsilon(x,\xi) = \rho_\varepsilon(x,\xi)K(x,\xi)$$

then K^ε has n-1 bounded derivatives and

$$|g_{\mu\nu}(K) - g_{\mu\nu}(K^\varepsilon)| \le \frac{1}{\ell}\int_0^\ell\int_0^\ell |K(s,t) - K^\varepsilon(s,t)|\,ds\,dt.$$

This together with Egorov's theorem shows that $g_{\mu\nu}(K^{\epsilon})$ converges to $g_{\mu\nu}(K)$ as ϵ tends to 0.

Since $K^{\epsilon}(s,t)$ is periodic of period ℓ, we obtain by integration by parts that

$$g_{\mu\nu}(K^{\epsilon}) = \left(\frac{\ell}{2\pi i}\right)^{n-1} \nu^{-(n-1)} g_{\mu\nu}(D_t^{n-1}K^{\epsilon})$$

provided $D_t^{n-1} K \not\equiv 0$. Note that $D_t^{n-1} K^{\epsilon} \not\equiv 0$ for small enough ϵ when $D_t^{n-1} K \not\equiv 0$, and in case $D_t^{n-1} K \equiv 0$ we have K is a polynomial in t of degree $\leq n-2$. However, since K is of the form $K(x,\xi) = \rho(x)\rho(\xi)G(x,\xi)$ where G is a Green's function and $\rho \in C^n[0,\ell]$, K has a discontinuity in its $(n-1)^{st}$ derivative. By the same argument as above $g_{\mu\nu}(D_t^{n-1} K^{\epsilon})$ converges to $g_{\mu\nu}(D_t^{n-1} K)$ as ϵ tends to 0. Thus for $D_t^{n-1} K \not\equiv 0$ we have

$$g_{\mu\nu}(K) = \left(\frac{\ell}{2\pi i}\right)^{n-1} \nu^{-(n-1)}\, g_{\mu\nu}(D_t^{n-1} K). \tag{3.6}$$

Let

$$\psi_m(s,t) = \begin{cases} 1 & 0 \leq t \leq s - \frac{1}{m}, \quad s + \frac{1}{m} \leq t \leq \ell \\ 1 - e^{\frac{(s-1/2m-t)^2}{2(s-1/m-t)}}, & s - \frac{1}{m} < t \leq s - \frac{1}{2m} \\ 0 & s - \frac{1}{2m} \leq t \leq s + \frac{1}{2m} \\ 1 - e^{\frac{(s+1/2m-t)^2}{2(t-1/m-s)}}, & s + \frac{1}{2m} \leq t < s + \frac{1}{m}\,. \end{cases}$$

Then $\psi_m(s,t)$ is infinitely differentiable with $|\psi_m| \le 1$.

Now consider

$$\ell\, g_{\mu\nu}(\psi_m F) = \int_0^\ell ds \int_0^\ell dt\ \psi_m(s,t)\, F(s,t) e^{+\frac{2\pi i}{\ell}(\mu s - \nu t)}$$

where

(3.7) $$F(s,t) = D_t^{n-1} K(s,t).$$

Then

$$\ell\, g_{\mu\nu}(\psi_m F) = \int_0^\ell ds \Big[\int_0^{s-\frac{1}{2m}} + \int_{s+\frac{1}{2m}}^{\ell} \Big]\Big[\psi_m(s,t)\, F(s,t) e^{+\frac{2\pi i}{\ell}(\mu s - \nu t)}\, dt\Big].$$

Note that

$$\left|\int_0^{s-\frac{1}{2m}} dt\ \psi_m(s,t) F(s,t) e^{-\frac{2\pi i}{\ell}\nu t}\right| = \left| -\frac{\ell}{2\pi i\nu}\, \psi_m(s,t)\, F(s,t) e^{-\frac{2\pi i}{\ell}\nu t}\Big|_{t=0}^{s-\frac{1}{2m}} \right.$$

$$\left. + \frac{\ell}{2\pi i \nu}\int_0^{s-\frac{1}{2m}} D_t(\psi_m(s,t)\, F(s,t)) e^{-\frac{2\pi i}{\ell}\nu t} \right|$$

$$\le \frac{\ell}{2\pi\nu}\Big[|F(s,0)| + \ell \sup_t B(s,t) \Big]$$

since $\psi(s, s-\frac{1}{2m}) = 0$ and

$$|D_t[\psi_m(s,t) F(s,t)]| = |F(s,t) D_t\psi_m(s,t) + \psi_m(s,t) D_t F(s,t)| \quad \text{for } t \text{ on } [0, s-\tfrac{1}{2m}]$$

$$\le |F(s,t)| + |D_t F(s,t)|\ ,$$

where use has been made of the fact that $D_t\psi_m(s,t) \le 1$ for $0 \le t \le s - \frac{1}{2m}$, moreover

$$|D_t[\psi_m(s,t) F(s,t)]| \le B_0(s,t)$$

where

$$B_0(s,t) \equiv |F(s,t)| + |\rho(s)| \left| \sum_{k=1}^{n} \Big[\binom{n}{k} D_t^k \rho(t) - \rho(t) p_n^{-1}(t) p_{n-k}(t)\Big] D_t^{n-k} G(s,t) \right|.$$

Here use has been made of Leibnitz rule and the fact that G satisfies the differential equation. Use of Leibnitz rule for F and the triangle inequality gives $B(s,t)$. Similarly

$$\left|\int_{s+\frac{1}{2\pi}}^{\ell} dt\, \psi_m(s,t)\, F(s,t) e^{-\frac{2\pi i}{\ell}\nu t}\right| \le \frac{\ell}{2\pi\nu}\left[|F(s,\ell)| + \ell \sup_t B(s,t)\right].$$

Thus

$$|\ell\, g_{\mu\nu}(\psi_m F)| \le \frac{\ell}{2\pi\nu}\int_0^{\ell}\left[\,|F(s,0)| + |F(s,\ell)| + 2\ell \sup_t B(s,t)\right] ds .$$

But

$$\ell|g_{\mu\nu}(\psi_m F) - g_{\mu\nu}(F)| \le \int_0^{\ell} ds \int_0^{\ell} |\psi_m(s,t) - 1|\,|F(s,t)|\,dt$$

$$\le \sup_{s,t} |F(s,t)| \int_0^{\ell} ds \int_0^{\ell} |\psi_m(s,t)-1|\,dt$$

and thus since ψ_m converges to 1 in L_1, means we get that

$$\ell|g_{\mu\nu}(F)| \le \frac{\ell}{2\pi\nu}\, C$$

and thus from (3.6) and (3.7)

$$\ell|g_{\mu\nu}(K)| \le \left(\frac{\ell}{2\pi}\right)^n C\,\frac{1}{\nu^n}$$

which is the result.

We are now in a position to state the general

<u>Theorem</u> 4. <u>Let</u>

$$g_{\mu\nu} = \frac{1}{\ell}\int_0^{\ell} ds \int_0^{\ell} dt\; K(s,t) e^{+i\frac{2\pi}{\ell}(\mu s - \nu t)}$$

and consider the matrix $(g_{\mu\nu})$; $\mu,\nu = 0, -1, 1, -2, 2, \ldots, \ell$, *where* $\ell = -\frac{m-1}{2}$ *if* m *is odd and* $-\frac{m}{2}$ *if* m *is even. If* $f_1, f_2, \ldots, f_m$ *are the eigenvalues of* $(g_{\mu\nu})$ *then the eigenvalues of* (3.1) *lie in*

$$\bigcup_{k=1}^{m} \{z\colon |z - f_k| < r_m\} \cup \{z\colon |z| < r_o\} \tag{3.8}$$

where

$$r_m^2 = 2m\,p_m^2, \quad r_0^2 = 2m\,p_m^2 \tag{3.9}$$

with

$$p_m^2 = m A^2 \left[\frac{(2\pi)^{2n}}{[2n]!} B_n - \left|\frac{m}{2}\right|^{-2n} - 2 \sum_{\nu=1}^{m/2-1} |\nu|^{-2n} \right] \tag{3.10}$$

for m *even and*

$$p_m^2 = m A^2 \left[\frac{(2\pi)^{2n}}{[2n]!} B_n - 2 \sum_{\nu=1}^{(m-1)/2} |\nu|^{-2n} \right] \tag{3.11}$$

for m *odd.* B_n *is the* n^{th} *Bernoulli number* ($B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$), *and* A *is given by* (3.3) *of lemma* 3.1. *Moreover, each component of* (3.8) *contains exactly as many eigenvalues as circles, where the eigenvalues and circles are counted with their multiplicities.*

Remark: It is a remarkable fact that the radii of the circles appearing in (3.8) depend only on the order of the differential equation and A.

Proof. Since $(g_{\mu\nu})$ is symmetric, there exists an orthogonal matrix T such that

$$T'(g_{\mu\nu})T = F$$

where F is a diagonal matrix with eigenvalues of $(g_{\mu\nu})$ down the diagonal and T' is the transpose of T. Thus we can take $t_0 = 1$ in theorem 3.

Let

$$\{k\} = [\tfrac{k}{2}](-1)^{k+1}, \qquad [\ell] \text{ denotes the greatest integer } \le \ell.$$

From theorem 3 we obtain

$$p_m^2 = \sum_{\nu=m+1}^{\infty} \sum_{\mu=1}^{m} |g_{\{\mu\},\{\nu\}}|^2$$

$$= \sum_{\nu=m+1}^{\infty} \sum_{\mu=2}^{m} |g_{\{\mu\},\{\nu\}}|^2 + \sum_{\nu=m+1}^{\infty} |g_{0,\{\nu\}}|^2 .$$

But by lemma (3.1)

$$|g_{\{\mu\},\{\nu\}}|^2 \le A^2 \{\nu\}^{-2n} ,$$

where A is given by lemma 3.1. We must now consider the cases where m is even and where m is odd.

<u>Case 1.</u> (m even) Here we have $\{m+1\} = \frac{m}{2}$ and thus

$$p_m^2 \le A^2 \sum_{\nu=m+1}^{\infty} \sum_{\mu=1}^{m} |\{\nu\}|^{-2n}$$

$$\le mA^2 \Big[2 \sum_{\nu=m/2+1}^{\infty} |\nu|^{-2n} + |\tfrac{m}{2}|^{-2n} \Big]$$

$$\le mA^2 \Big[2 \sum_{\nu=1}^{\infty} |\nu|^{-2n} - |\tfrac{m}{2}|^{-2n} - 2 \sum_{\nu=1}^{m/2-1} |\nu|^{-2n} \Big]$$

$$\le mA^2 \Big[\frac{2^{2n}\pi^{2n}}{[2n]!} B_n - |\tfrac{m}{2}|^{-2n} - 2 \sum_{\nu=1}^{m/2-1} |\nu|^{-2n} \Big] ,$$

where B_n denotes the n^{th} Bernoulli number.

Case 2. (m odd) Similarly we obtain

$$p_m^2 \le m A^2 \left[\frac{(2\pi)^{2n}}{[2n]!} B_n - 2 \sum_{\nu=1}^{m-1/2} |\nu|^{-2n} \right].$$

§4. Numerical examples. Since the radii r_m of the circles in Theorem 4 depend only on the order of the differential equation and the constant A we give below a table for r_m/A for $n=2$ and $n=4$.

(4.1)

m	r_m/A, $n=2$	r_m/A, $n=4$
4	1.80793	.366
6	1.40176	.117
8	1.18939	.0532
10	1.053	.02937
20	.734	.00489
30	.598	.00175
40	.517	.000850
50	.462	.000486

Next we apply the theory to several examples. As a first illustration we consider a whirling shaft of length 2π see e.g., [7, p.443] that rotates between bearings at 0 and 2π. The differential equation governing the transverse displacement u is of the form

$$u^{(4)}(x) = \lambda u$$

with the boundary conditions $u(0) = u(2\pi) = u''(0) = u''(2\pi) = 0$.

The Green's function and also the kernel of the integral equation for this problem is given by

$$G(x,\xi) = \begin{cases} \frac{\xi - 2\pi}{12\pi}\,[\xi(\xi - 4\pi)x + x^3] & x < \xi \\ \frac{x - 2\pi}{12\pi}\,[x(x - 4\pi)\xi + \xi^3] & x \geq \xi \,, \end{cases}$$

and a bound for A is $2 + 4\pi$. Combining this with table (4.1) we get

$$(4.2) \qquad \begin{aligned} A &= 2 + 4\pi \\ r_4 &= 5.33129 \\ r_6 &= 1.70426 \\ r_8 &= .77493 \\ r_{10} &= .427814 \\ r_{20} &= .065388 \\ r_{30} &= .0254911 \\ r_{40} &= .0123814 \\ r_{50} &= .0070792 \end{aligned}$$

The computed eigenvalues, using the Green's function, are compared with the exact eigenvalues in table (4.6). The routine for finding the eigenvalues of the matrix was only accurate to three places.

As a second illustration we consider an inhomogeneous string, fixed at one end and restricted to move transversally under an elastic force at the other end. The differential equation governing the transverse displacement y is of the form

$$-y'' = \lambda(2-x^2)y, \quad y(0) = 0, \quad 2y(1) + y'(1) = 0$$

see [3, p.252]. By differentiation and because of uniqueness this problem is equivalent to solving:

$$[(2-x^2)^{-1}y'']'' = \lambda^2(2-x^2)y$$

$$y(0) = y''(0) = 0, \quad 2y(1) + y'(1) = 0, \quad 4y''(1) + y'''(1) = 0.$$

The Green's function for this problem is given by

$$(4.3) \quad G(x,\xi) = \frac{2}{3}\Big[\xi\ell(x) + x\ell(\xi)\Big] + \frac{7}{15}\Big(\frac{2}{3}\xi - 1\Big)\Big(\frac{2}{3}x - 1\Big) - \ell(x) - xm(\xi), \quad x < \xi$$

$$= \frac{2}{3}\Big[x\ell(\xi) + \xi\ell(x)\Big] + \frac{7}{15}\Big(\frac{2}{3}x - 1\Big)\Big(\frac{2}{3}\xi - 1\Big) - \ell(\xi) - \xi m(x), \quad x \geq \xi$$

where

$$\ell(x) = \frac{1}{60}\,[-3x^5 + 20x^3 - 45x + 28]$$

$$m(x) = \frac{1}{12}\,[-x^4 + 12x^2 - 20x + 9]\,.$$

The kernel $K(x,\xi)$ of the integral equation is given by

$$(4.4) \qquad K(x,\xi) = \sqrt{2 - x^2}\;\sqrt{2 - \xi^2}\;\; G(x,\xi)$$

and the entries in the equivalent matrix are given by

$$g_{\mu\nu} = \int_0^1 ds \int_0^1 dt\, e^{+2\pi i(\mu s - \nu t)}\, K(s,t).$$

A bound for A is 5. Combining this with table (4.1) we get

$$(4.5) \qquad \begin{aligned} A &= 5 \\ r_4 &= 1.83 \\ r_6 &= .585 \\ r_8 &= .266 \\ r_{10} &= .14685 \\ r_{20} &= .022445 \\ r_{30} &= .00875 \\ r_{40} &= .00425 \\ r_{50} &= .00243 \end{aligned}$$

The computed eigenvalues, using the Green's function, are compared with the estimates obtained by Collatz [3, p.257] in table (4.7). In this case, the routine for finding the eigenvalues of the matrix is accurate to three decimals.

(4.6)

<u>Twirling Shaft</u>

	Eigenvalues computed via Green's function	Exact eigenvalues	Actual Error	Theoretical error estimate for $A = 2+4\pi$
λ_1	15.99950	16.00000	.000050	.0124
λ_2	.9999758	1.00000	.000025	.0124
λ_3	.1975536	.197531	.000000	⋮
λ_4	.0624986	.062500	.000002	
λ_5	.0255906	.025600	.000009	
λ_6	.0123447	.0123457	.000001	
λ_7	.0066600	.0066639	.000003	
λ_8	.0039062	.0039062	.0	
λ_9	.0024354	.002414	.000021	
λ_{10}	.0015999	.001600	.000001	

(4.7)

Eigenvalues	Reciprocal squared eigenvalues computed via Green's Function	Theoretical Error	Computed error range	Computed reciprocal eigenvalues	Reciprocal eigenvalues via Collatz
λ_1	.089252	.00425	$.085002 \le \lambda_1^{-2} \le .093502$	$.291551 \le \lambda_1^{-1} \le .30578$	$.254 \le \lambda_1^{-1} \le .308$
λ_2	.003763	.00425	$-.00049 \le \lambda_2^{-2} \le .00813$	$-.02207 \le \lambda_2^{-1} \le .08952$	$.0539 \le \lambda_2^{-1} \le .0656$
λ_3	.0006098	.00425	$-.00364 \le \lambda_3^{-2} \le .00486$	$-.0603 \le \lambda_3^{-1} \le .0697$	$.0216 \le \lambda_3^{-1} \le .0264$
λ_4	.00017066	.00425	$-.00408 \le \lambda_4^{-2} \le .00442$	$-.0639 \le \lambda_4^{-1} \le .0665$	
λ_5	.0000647	.00425	$-.00418 \le \lambda_5^{-2} \le .00431$	$-.0646 \le \lambda_5^{-1} \le .0657$	